Suppose we know that ψ_p is a particular soln of this equation and ψ is another soln

$$L[\psi-\psi_p] = L[\psi] - L[\psi_p] = b-b=0 \quad \text{on } I$$

This shows that $\psi-\psi_p$ is a soln of the homogonous equation $L(y)=0$

$\therefore$ If φ_1, φ_2 are L.I solns of $L(y)=0$

There are unique constant c_1, c_2 suchthat

$$\psi - \psi_p = c_1\varphi_1 + c_2\varphi_2$$

In otherwords every soln ψ of

$L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2.$$

Problems:

1(a) $y''-y'-2y=0$ find all soln of the following equation

Soln:

Given $y''-y'-2y=0 \rightarrow ①$

It is of the form

$y''+a_1y'+a_2y=0 \rightarrow ②$

① & ② compare

$a_1=-1, \quad a_2=-2$

Characteristic polynomial is

$p(r) = r^2+a_1r+a_2$

$p(r) = r^2-r-2$

To find solns

put $p(r)=0$

$r^2-r-2=0$

$(r-2)(r+1)=0$

$\Rightarrow r = 2, -1$

Every soln ϕ of ② is of the form

$\phi = c_1e^{r_1x}+c_2e^{r_2x}$

$\phi = c_1e^{-x}+c_2e^{2x}$ is the soln of ①

where c_1, c_2 are constants

1.b) Solve the harmonic oscillator equation $y''+\omega^2 y=0$

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Soln:

Given $y''+\omega^2 y=0 \rightarrow (1)$

It is of the form

$y''+a_1y'+a_2y=0 \rightarrow (2)$

compare (1) & (2)

$a_1=0, \quad a_2=\omega^2$

The characteristic polynomial is

$p(r)=r^2+a_1r+a_2$

$=r^2+\omega^2$

To find soln

put $p(r)=0$

$r^2+\omega^2=0$

$r^2=-\omega^2$

$r=\pm i\omega$

Every soln ϕ is of the form

$\phi(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

$\phi(x)=c_1\cos\omega x+c_2\sin\omega x$

(or) $\phi(x)=c_1e^{i\omega x}+c_2e^{-i\omega x}$

Where c_1, c_2 are constants

Note:

Take $c_1=1/2, \quad c_2=1/2$ in eqn (3)

Case (i)

$$\phi(x)=\frac{e^{i\omega x}+e^{-i\omega x}}{2}$$

$\phi(x)=\cos\omega x$

Case (ii)

Take $c_1=\frac{1}{2i}, \quad c_2=\frac{-1}{2i}$

$$\phi(x)=\frac{e^{i\omega x}-e^{-i\omega x}}{2i}$$

$\phi(x)=\sin\omega x$

1.c) Solve the harmonic oscillator equation $y''-4y'+5y=0$

Soln: Given equation $y''-4y'+5y=0 \rightarrow (1)$

It is the form

$y''+a_1y'+a_2y=0 \rightarrow (2)$

compare (1) & (2)

$a_1=-4, \quad a_2=5$

The characteristic polynomial is

$p(r) = r^2 + a_1 r + a_2$

$= r^2 - 4r + 5$

$a=1,\ b=-4,\ c=5$

$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{4 \pm \sqrt{16-4(5)}}{2}$

$= \frac{4 \pm \sqrt{-4}}{2}$

$= \frac{4 \pm 2i}{2}$

$r = 2 \pm i$

To find soln

put $p(r) = 0$

$r^2 - 4r + 5 = 0$

$\therefore r = 2 \pm i$

Every soln of φ is of the form

$\varphi(x) = c_1 e^{(2+i)x} + c_2 e^{(2-i)x}$

(or)

$\varphi(x) = e^{2x}[c_1 \cos x + c_2 \sin x]$

1.d) $y'' + 16y = 0$

Soln:

Given equation $y'' + 16y = 0 \rightarrow (1)$

It is of the form

$y'' + a_1 y' + a_2 y = 0 \rightarrow (2)$

compare (1) & (2)

$a_1 = 0,\ a_2 = 16$

The characteristic polynomial is

$p(r) = r^2 + a_1 r + a_2$

$p(r) = r^2 + 16$

Put $p(r) = 0$

$r^2 + 16 = 0$

$r^2 = -16$

$r = \pm 4i$

Every soln of φ is of the form

$\varphi(x) = c_1 e^{i4x} + c_2 e^{-i4x}$

(or)

$\varphi(x) = c_1 \cos 4x + c_2 \sin 4x$

1.e) $y'' + 2iy' + y = 0$

Soln:

Given equation $y'' + 2iy' + y = 0 \rightarrow (1)$

It is the form $y'' + a_1 y' + a_2 y = 0 \rightarrow (2)$

compare (1) & (2)

$a_1 = 2i,\ a_2 = 1$

The characteristic polynomial

$$p(r) = r^2 + a_1 r + a_2$$

$$p(r) = r^2 + 2ir + 1$$

$$p(r) = 0$$

$$r^2 + 2ir + 1 = 0$$

$$r = i(-1 \pm \sqrt{2})$$

$a = 1, \quad b = 2i, \quad c = 1$

$$r = \frac{-2i \pm \sqrt{4i^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-2i \pm \sqrt{-4-4}}{2}$$

$$= \frac{-2i \pm \sqrt{-8}}{2}$$

$$= \frac{-2i \pm i2\sqrt{2}}{2}$$

$$= -i \pm i\sqrt{2}$$

$$= i(-1 \pm \sqrt{2})$$

Every soln ϕ is the form

$$\phi(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$\phi(x) = C_1 e^{i(-1+\sqrt{2})x} + C_2 e^{i(-1-\sqrt{2})x}$$

Where C_1, C_2 are constants

1.f) $y'' - 4y = 0$

Soln:

Given equation $\quad y'' - 4y = 0 \rightarrow (1)$

It is of the form $\quad y'' + a_1 y' + a_2 y = 0 \rightarrow (2)$

Compare (1) & (2) $\quad a_1 = 0, \quad a_2 = -4$

The characteristic polynomial

$$p(r) = r^2 + a_1 r + a_2$$

$$p(r) = r^2 - 4$$

$$p(r) = 0$$

$$r^2 - 4 = 0$$

$$r^2 = 4$$

$$r = \pm 2$$

Every soln ϕ is of the form

$$\phi(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$\phi(x) = C_1 e^{2x} + C_2 e^{-2x}$$

Where C_1, C_2 are constants

1.g) $y'' + (3i-1)y' - 3iy = 0$

Soln:

Given equation $\quad y'' + (3i-1)y' - 3iy = 0 \rightarrow (1)$

It is of the form $\quad y'' + a_1 y' + a_2 y = 0 \rightarrow (2)$

$$p(r) = r^2 + a_1 r + a_2$$

$$p(r) = r^2 + (3i-1)r - 3i$$

$$p(r) = 0$$

$$r^2 + (3i-1)r - 3i = 0$$

$$r^2 + 3ir - r - 3i = 0$$

$$r^2 - r + 3ir - 3i = 0$$

$r(r-1)+3i(r-1)=0$

$(r+3i)(r-1)=0$

$r=1,-3i$

Every soln φ is of the form

$\varphi(x)=c_1e^{r_1x}+c_2e^{r_2x}$

$\varphi(x)=c_1e^{x}+c_2e^{-3ix}$

Where c_1, c_2 are constants

1.h) $3y''+2y'=0$

Soln:

Given equation $3y''+2y'=0 \rightarrow ①$

It is of the form $y''+a_1y'+a_2y=0 \rightarrow ②$

from ① & ②

$a_1=2,\ a_2=0$

The characteristic polynomial

$p(r)=r^2+a_1r+a_2$

$p(r)=3r^2+2r$

$p(r)=0$

$3r^2+2r=0$

$r(3r+2)=0$

$r=0,\quad 3r+2=0$

$3r=-2$

$r=-2/3$

Every soln φ is of the form

$\varphi(x)=c_1e^{r_1x}+c_2e^{r_2x}$

$=c_1e^{0x}+c_2e^{-2/3x}$

$\varphi(x)=c_1+c_2e^{-2/3x}$

Where c_1, c_2 are constants

1, i) $y''=0$

Soln:

Given equation $y''=0 \rightarrow (1)$

It is of the form $y''+a_1y'+a_2y=0 \rightarrow (2)$

The characteristic polynomial

$p(r) = r^2+a_1r+a_2$

$p(r) = r^2$

$p(r)=0$

$r^2=0$

$r=0$

Every soln φ is of the form

$\varphi(x) = (c_1+c_2x)e^{r_1x} = (c_1+c_2x)e^{0x}$

$\varphi(x) = c_1+c_2x$

where c_1, c_2 are constants.

2.a) consider the equation $y''+y'-6y=0$

(i) compute the soln φ satisfying $\varphi(0)=1$, $\varphi'(0)=0$

(ii) compute the soln ψ satisfying $\psi(0)=0$, $\psi'(0)=1$

(iii) compute $\varphi(1)$ & $\psi(1)$

Soln:

Given equation $y''+y'-6y=0 \rightarrow (1)$

It is of the form $y''+a_1y'+a_2y=0 \rightarrow (2)$

The characteristic polynomial

$p(r)=r^2+a_1r+a_2$

$p(r)= r^2+r-6$

$p(r)=0$

$r^2+r-6=0$

$(r-2)(r+3)=0$

$r= 2,-3$

Every soln φ is of the form

$\varphi(x)= c_1e^{r_1x}+c_2e^{r_2x}$

$\varphi(x)= c_1e^{2x}+c_2e^{-3x} \rightarrow (3)$

$\varphi'(x) = 2c_1e^{2x}+(-3c_2e^{-3x})$

$\varphi'(x)= 2c_1e^{2x}-3c_2e^{-3x}$

Given $\varphi(0)=1$

i.e) Sub $x=0$ in (3)

$$\varphi(0)=c_1e^0+c_2e^0=1$$

$$c_1+c_2=1 \rightarrow (5)$$

Given $\varphi'(0)=0$

i.e) Sub $x=0$ in (4)

$$\varphi'(0)=2c_1-3c_2=0$$

$$2c_1-3c_2=0 \rightarrow (6)$$

Solve (5) and (6)

$$(5)\times 3 \Rightarrow 3c_1+3c_2=3$$

$$(6) \Rightarrow \underline{2c_1-3c_2=0}$$

$$5c_1=3$$

$$c_1=3/5$$

$c_1=3/5$ Sub in (5)

$$c_2=2/5$$

$$\therefore \varphi(x)=\frac{3}{5}e^{2x}+\frac{2}{5}e^{-3x} \rightarrow (7)$$

(ii) To find ψ

Let $\psi(x)=c_1e^{2x}+c_2e^{-3x} \rightarrow (8)$

$$\psi'(x)=2c_1e^{2x}-3c_2e^{-3x} \rightarrow (9)$$

Given $\psi(0)=0$

i.e) Sub $x=0$ in (8)

$$\psi(0)=c_1+c_2=0$$

$$\Rightarrow c_1+c_2=0 \rightarrow (10)$$

Given $\psi'(0)=1$ i.e) Sub $x=0$ in (9)

$$\psi'(0)=2c_1-3c_2=1$$

$$\Rightarrow 2c_1-3c_2=1 \rightarrow (11)$$

$$(10)\times 3 \Rightarrow 3c_1+3c_2=0$$

$$(11) \Rightarrow \underline{2c_1-3c_2=1}$$

$$5c_1=1$$

$$c_1=1/5$$

$c_1 = 1/5$ Sub in (10)

$c_2 = -1/5$

$\psi(x) = \frac{1}{5} e^{2x} - \frac{1}{5} e^{-3x} \rightarrow$ (12)

(iii) To find $\varphi(1)$ and $\psi(1)$

put $x=1$ in (11) and (12)

$\varphi(1) = \frac{3}{5} e^{2} + \frac{2}{5} e^{-3}$

$\psi(1) = \frac{1}{5} e^{2} - \frac{1}{5} e^{-3}$

2.b) Find all Soln φ of $y''+y=0$ Satisfying

i) $\varphi(0)=1$, $\varphi(\pi/2)=2$

(ii) $\varphi(0)=0$, $\varphi(\pi)=0$

(iii) $\varphi(0)=0$. $\varphi'(\pi/2)=0$

(iv) $\varphi(0)=0$, $\varphi(\pi/2)=0$

Soln:

Given equation $y''+y=0$

It is the form $y''+a_1y'+a_2y=0$

The characteristic polynomial

$p(r) = r^2+1$

$p(r)=0$

$r^2+1=0$

$r^2=-1$

$r=\pm i$

Every Soln φ is of the form

$\varphi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$\varphi(x) = c_1 e^{ix} + c_2 e^{-ix}$

(or)

$\varphi(x) = c_1 \cos x + c_2 \sin x \rightarrow$ (1)

(i) Given $\varphi(0)=1$

$\varphi(0) = c_1 \cos 0 + c_2 \sin 0$

$c_1 = 1$

Given $\varphi(\pi/2) = 2$

$\varphi(\pi/2) = c_1 \cos \pi/2 + c_2 \sin \pi/2$

$2 = c_2$

$c_2 = 2$ $\therefore \varphi(x) = \cos x + 2 \sin x$.

(ii) $\varphi(0)=0$

$\varphi(0) = c_1 \cos 0 + c_2 \sin 0$

$c_1 = 0$

Given $\varphi(\pi)=0$

$\varphi(\pi) = c_1 \cos\pi + c_2 \sin\pi$

$0 = -c_1$

$c_1 = 0$

$\therefore \varphi(x) = c_2 \sin x$

$\varphi(x) = c \sin x$

$\therefore$ c is any constant

(iii) Given $\varphi(0)=0$ by (ii)

$\therefore c_1 = 0$

$\varphi'(x) = -c_1 \sin x + c_2 \cos x$

$\varphi'(\pi/2) = -c_1 \sin \pi/2 + c_2 \cos \pi/2$

$0 = -c_1$

$c_1 = 0$

$\therefore \varphi(x) = c_2 \sin x$

$\varphi(x) = c \sin x$

$\therefore$ c is any constant

(iv) $\varphi(0)=0$ by (ii)

$c_1 = 0$

Given $\varphi(\pi/2) = 0$

$\varphi(\pi/2) = c_1 \cos \pi/2 + c_2 \sin \pi/2$

$c_2 = 0$

$\therefore \varphi(x) = 0 \cos x + 0 \sin x$

$\varphi(x) = 0.$

2.c) let φ be a soln of the equation $y'' + a_1 y' + a_2 y = 0$ where a_1, a_2 are constants. If $\psi(x) = e^{(\frac{a_1}{2})x} \varphi(x)$ S.T ψ satisfies an equation $y'' + ky = 0$ where k is some constant compute k.

Soln:

Given φ is the soln of

$$y''+a_1y'+a_2y=0 \to (1)$$

$$\varphi''+a_1\varphi'+a_2\varphi=0 \to (2)$$

Also given $\psi(x)=e^{(a_1/2)x}\varphi(x) \to (3)$

$$\Rightarrow \varphi(x)=e^{(-a_1/2)x}\psi(x) \to (4)$$

Sub (4) in (2)

$$(e^{(-a_1/2)x}\psi)''+a_1(e^{-a_1/2x}\psi)'+a_2(e^{-a_1/2x}\psi)=0 \to (5)$$

$$(e^{-a_1/2x}\psi)'=-\frac{a_1}{2}e^{-\frac{a_1}{2}x}\psi+e^{-a_1/2x}\psi' \to (6)$$

$$(e^{-a_1/2x}\psi)''=\frac{a_1^2}{2}e^{-a_1/2x}\psi-\frac{a_1}{2}e^{-\frac{a_1}{2}x}\psi'-\frac{a_1}{2}e^{-\frac{a_1}{2}x}\psi'+e^{-a_1/2x}\psi'' \to (7)$$

Sub (6) & (7) in (5)

$$\left\{\left[\frac{a_1^2}{4}e^{-\frac{a_1}{2}x}\psi-\frac{a_1}{2}e^{-\frac{a_1}{2}x}\psi'-\frac{a_1}{2}e^{-\frac{a_1}{2}x}\psi'+e^{-\frac{a_1}{2}x}\psi''\right]+ a_1\left[-\frac{a_1}{2}e^{-\frac{a_1}{2}x}\psi+e^{-\frac{a_1}{2}x}\psi'\right]+a_2\left[e^{-\frac{a_1}{2}x}\psi\right]\right\}=0$$

$$e^{-\frac{a_1}{2}x}\left[\frac{a_1^2}{4}\psi-\frac{a_1}{2}\psi'-\frac{a_1}{2}\psi'+\psi''-\frac{a_1^2}{2}\psi+a_1\psi'+a_2\psi\right]=0$$

$$\frac{a_1^2}{4}\psi-\frac{a_1^2}{2}\psi+a_1\psi'-a_1\psi'+a_2\psi+\psi''=0$$

$$-\frac{a_1^2}{2}\psi+a_2\psi+\psi''=0$$

$$\psi''+\psi\left(a_2-\frac{a_1^2}{2}\right)=0$$

$$\psi''+K\psi=0$$

where $K=a_2-\frac{a_1^2}{2}$

$$\therefore y''+Ky=0$$

2.d) Consider the equation $y''+K^2y=0$ where K is non-negative constant.

A) for what values of K will there exist non-trivial solns φ satisfying

(i) $\varphi(0)=0$, $\varphi(\pi)=0$

(ii) $\varphi'(0)=0$, $\varphi'(\pi)=0$

(iii) $\varphi(0)=\varphi(\pi)$, $\varphi'(0)=\varphi'(\pi)$

(iv) $\varphi(0)=-\varphi(\pi)$, $\varphi'(0)=-\varphi'(\pi)$

B) Find out the non-trivial solns for each of the cases (i)-(iv) in (a)

Soln:

A) Given equations $y'' + k^2 y = 0 \rightarrow (1)$

The characteristic polynomial

$P(r) = r^2 + k^2$

$r^2 + k^2 = 0$

$r^2 = -k^2$

$r = \pm ik$

$\phi(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\phi(x) = e^{0x}(c_1 \cos kx + c_2 \sin kx)$

$\phi(x) = c_1 \cos kx + c_2 \sin kx \rightarrow (2)$

(i) Given $\phi(0) = 0$

$\phi(0) = c_1 \cos k(0) + c_2 \sin k(0)$

$c_1 = 0$

Given $\phi(\pi) = 0$

$\phi(\pi) = c_1 \cos k\pi + c_2 \sin k\pi$

$c_1 = 0$

$\therefore \phi(x) = 0 \cos kx + c_2 \sin kx$

$\therefore \phi(x) = c \sin kx$

$\therefore$ c is any constant

(ii) $\phi'(x) = -c_1 k \sin kx + c_2 k \cos kx \rightarrow (3)$

Given $\phi'(0) = 0$

$\phi'(0) = -c_1 k \sin k(0) + c_2 k \cos k(0)$

$c_2 k = 0$

$c_2 = 0$

Given $\phi'(\pi) = 0$

$\phi'(\pi) = -c_1 k \sin \pi + c_2 k \cos k\pi$

$c_2 k = 0$

$c_2 = 0$

$\therefore \phi(x) = c_1 \cos kx$

$\phi(x) = c \cos kx$

c is any constant

(iii) $\varphi(0) = \varphi(\pi)$

$\Rightarrow C_1 = 0, \ C_1 = 0$

$\therefore \varphi(0) = \varphi(\pi)$

(iv) $\varphi'(0) = \varphi'(\pi)$

$\varphi'(0) \Rightarrow C_2 = 0, \ \varphi'(\pi) \Rightarrow C_2 = 0$

$\therefore \varphi'(0) = \varphi'(\pi)$

(v) $\varphi(0) \Rightarrow C_1 = 0$

$-\varphi(\pi) \Rightarrow -C_1 = 0$

$\therefore \varphi(0) = -\varphi(\pi)$

3.a) Find the solns of the following initial value problem

$y'' - 2y' - 3y = 0, \quad y(0) = 0, \ y'(0) = 1$

Soln:

Given equation is

$y'' - 2y' - 3y = 0$

The characteristic polynomial

$p(r) = r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3, -1$

The soln is

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$y(x) = C_1 e^{3x} + C_2 e^{-x} \rightarrow (1)$

Given $y(0) = 0$

$\therefore y(0) = C_1 e^0 + C_2 e^0$

$C_1 + C_2 = 0 \rightarrow (2)$

$y'(x) = 3C_1 e^{3x} - C_2 e^{-x} \rightarrow (3)$

Given $y'(0) = 1$

$y'(0) = 3C_1 e^0 - C_2 e^0$

$3C_1 - C_2 = 1 \rightarrow (4)$

Solve (2) & (4)

$C_1 + C_2 = 0$

$3C_1 - C_2 = 1$

$4C_1 = 1$

$C_1 = 1/4$

$c_1 = \frac{1}{4}$ sub in ②

$c_2 = -1/4$

$\therefore y(x) = \frac{1}{4} e^{3x} - \frac{1}{4} e^{-x}$

$y(x) = \frac{1}{4} [e^{3x} - e^{-x}]$

3.b) $y'' + 10y = 0$, $y(0) = \pi$, $y'(0) = \pi^2$

Soln:

Given equation is $y'' + 10y = 0$

The characteristic polynomial

$r^2 + 10 = 0$

$r^2 = -10$

$r = \pm i\sqrt{10}$

$y(x) = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

$y(x) = c_1 \cos \sqrt{10} x + c_2 \sin \sqrt{10} x \rightarrow ①$

Given $y(0) = \pi$

$y(0) = c_1 \cos \sqrt{10}(0) + c_2 \sin \sqrt{10}(0)$

$c_1 = \pi$

$y'(x) = -\sqrt{10} c_1 \sin \sqrt{10} x + \sqrt{10} c_2 \cos \sqrt{10} x \rightarrow ②$

Given $y'(0) = \pi^2$

$y'(0) = -\sqrt{10} c_1 \sin \sqrt{10}(0) + \sqrt{10} c_2 \cos \sqrt{10}(0)$

$\pi^2 = \sqrt{10} c_2$

$c_2 = \frac{\pi^2}{\sqrt{10}}$

$\therefore y(x) = \pi \cos \sqrt{10} x + \frac{\pi^2}{\sqrt{10}} \sin \sqrt{10} x.$

4a) The forms φ_1, φ_2 defined
Determine whether they are linearly dependent (or) independent there.

a) $\varphi_1(x) = x$, $\varphi_2(x) = e^{rx}$, r is a complex constant.

Soln:

Given $\varphi_1(x) = x$, $\varphi_2(x) = e^{rx}$

W.K.T

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & e^{rx} \\ 1 & re^{rx} \end{vmatrix}$$

$$= xre^{rx} - e^{rx}$$

$$= (xr - 1)e^{rx} \neq 0$$

$\therefore$ r is complex constant

$\therefore$ The function φ_1, φ_2 are linearly independent.

4.b) $\varphi_1(x) = x$, $\varphi_2(x) = xe^x$

Soln:

Given $\varphi_1(x) = x$, $\varphi_2(x) = xe^x$

$\varphi_1'(x) = 1$, $\varphi_2'(x) = xe^x + e^x$

W.K.T

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix}$$

$$= x(xe^x + e^x) - xe^x$$

$$= x^2e^x + xe^x - xe^x$$

$$= x^2e^x \neq 0$$

$\therefore$ The function φ_1, φ_2 are linearly independent.

4.c) $\varphi_1(x) = x^2$, $\varphi_2(x) = 5x^2$

Soln: $\varphi_1(x) = x^2$, $\varphi_2 = 5x^2$

$\varphi_1'(x) = 2x$, $\varphi_2' = 10x$

W.K.T

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 5x^2 \\ 2x & 10x \end{vmatrix}$$

$$= 10x^3 - 10x^3$$

$$W(\varphi_1, \varphi_2)(x) = 0$$

$\therefore$ The function φ_1, φ_2 are linearly dependent.

4.d) $\varphi_1(x) = e^x \cos x$, $\varphi_2(x) = e^x \sin x$

Soln:

$\varphi_1(x) = e^x \cos x$ $\qquad$ $\varphi_2(x) = e^x \sin x$

$\varphi_1'(x) = e^x \cos x - e^x \sin x$ $\qquad$ $\varphi_2'(x) = e^x \sin x + e^x \cos x$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^x \cos x (e^x \sin x + e^x \cos x) - e^x \sin x (e^x \cos x - e^x \sin x)$$

$$= e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \sin x \cos x + e^{2x} \sin^2 x$$

$$= e^{2x} \cos^2 x + e^{2x} \sin^2 x$$

$$= e^{2x} (\cos^2 x + \sin^2 x)$$

$$W(\varphi_1, \varphi_2) = e^{2x} \neq 0$$

$\therefore$ The functions φ_1, φ_2 are linearly independent.

4.e) $\varphi_1(x) = x$, $\varphi_2(x) = |x|$

Soln:

Given $\varphi_1(x) = x$ $\qquad$, $\varphi_2(x) = |x| = \pm x$

$\varphi_1'(x) = 1$ $\qquad$ $\varphi_2'(x) = 1, -1$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} \quad , \quad W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & x \\ 1 & 1 \end{vmatrix} \qquad = \begin{vmatrix} x & -x \\ 1 & -1 \end{vmatrix}$$

$$= x - x \qquad = -x + x$$

$$= 0 \qquad = 0$$

$\therefore$ The function φ_1, φ_2 are linearly dependent

4.f) $\varphi_1(x) = \cos x$, $\varphi_2(x) = \sin x$

Soln:

Given $\varphi_1(x) = \cos x$, $\varphi_2(x) = \sin x$

$\varphi_1'(x) = -\sin x$, $\varphi_2'(x) = \cos x$

$$w(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1 \neq 0$$

$\therefore$ The functions φ_1, φ_2 are linearly independent.

4.g) $\varphi_1(x) = \sin x$, $\varphi_2(x) = e^{ix}$

Soln:

$\varphi_1(x) = \sin x$, $\varphi_2(x) = e^{ix}$

$\varphi_1'(x) = \cos x$, $\varphi_2'(x) = ie^{ix}$

$$w(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & e^{ix} \\ \cos x & ie^{ix} \end{vmatrix}$$

$$= ie^{ix} \sin x - e^{ix} \cos x$$

$$= e^{ix}(i\sin x - \cos x) \neq 0$$

$\therefore$ The functions φ_1, φ_2 are linearly independent

4.h) $\varphi_1(x) = \cos x$, $\varphi_2(x) = 3(e^{ix} + e^{-ix})$

Soln:

$\varphi_1(x) = \cos x$ $\qquad$ $\varphi_2(x) = 3(e^{ix} + e^{-ix})$

$\varphi_1'(x) = -\sin x$ $\qquad$ $\varphi_2'(x) = 3(ie^{ix} - ie^{-ix})$

$$w(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} \cos x & 3(e^{ix} + e^{-ix}) \\ -\sin x & 3(ie^{ix} - ie^{-ix}) \end{vmatrix}$$

$$= 3i(\cos x)(e^{ix} - e^{-ix}) + 3\sin x(e^{ix} + e^{-ix})$$

$$= 3i\cos x(2i\sin x) + 3\sin x(2\cos x)$$

$$= -6\cos x \sin x + 6\cos x \sin x$$

$$= 0$$

$\therefore$ The functions φ_1, φ_2 are linearly dependent.

4. i) $\varphi_1(x) = x^2$, $\varphi_2(x) = x|x|$, $-\infty < x < \infty$

(36)

Soln:

$\varphi_1(x) = x^2$, $\varphi_2(x) = x|x| \Rightarrow \pm x^2$

$\varphi_1'(x) = 2x$ $\qquad \varphi_2'(x) = \pm 2x$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix}$$

$$= 2x^3 - 2x^3$$

$$= 0$$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix}$$

$$= -2x^3 + 2x^3$$

$$= 0$$

The functions φ_1, φ_2 are linearly dependent.

5. a)

i) Let φ_n be any function satisfying the boundary value problem. $y'' + n^2 y = 0$, $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$

Where $n = 0, 1, 2 \ldots$ show that $\int_0^{2\pi} \varphi_n(x)\varphi_m(x)dx = 0$, if $n \neq m$

Show that $\cos nx$ and $\sin nx$ are functions satisfying in boundary value problem given in

(ii) Then implies that

$$\int_0^{2\pi} \cos nx \cos mx\, dx = 0, \qquad \int_0^{2\pi} \cos nx \sin mx\, dx = 0,$$

$$\int_0^{2\pi} \sin nx \sin mx\, dx = 0 \quad (n \neq m)$$

Soln:

Given equation is $y'' + n^2 y = 0 \rightarrow (1)$

Given φ_n satisfying eqn

$$\varphi_n'' + n^2 \varphi_n = 0$$

$$\varphi_n'' = -n^2 \varphi_n \rightarrow (2)$$

for $n \neq m$ $\quad \varphi_m'' = -m^2 \varphi_m \rightarrow (3)$

$(2) \times \varphi_m \Rightarrow \varphi_n'' \varphi_m = -n^2 \varphi_n \varphi_m$

$(3) \times \varphi_n \Rightarrow \underset{(-)}{\varphi_m''} \varphi_n = \underset{(+)}{-m^2} \varphi_n \varphi_m$

$$\varphi_n'' \varphi_m - \varphi_m'' \varphi_n = (m^2 - n^2)\varphi_n \varphi_m$$

$$\varphi_m'' \varphi_n - \varphi_n'' \varphi_m = (n^2 - m^2)\varphi_n \varphi_m \rightarrow (4)$$

Now, $[\varphi_m'\varphi_n - \varphi_n'\varphi_m]' = \varphi_m''\varphi_n + \varphi_m'\varphi_n' - \varphi_n''\varphi_m + \varphi_n'\varphi_m'$ (27)

$= \varphi_m''\varphi_n - \varphi_n''\varphi_m$

$[\varphi_m'\varphi_n - \varphi_n'\varphi_m]' = (n^2-m^2)\varphi_n\varphi_m \rightarrow (5)$

Integrating both sides from 0 to 2π in (5)

We get

$\varphi_m'\varphi_n - \varphi_n'\varphi_m = \int_0^{2\pi}(n^2-m^2)\varphi_n\varphi_m\,dx$

$\varphi_m'\varphi_n - \varphi_n'\varphi_m = (n^2-m^2)\int_0^{2\pi}\varphi_n\varphi_m\,dx$

$\varphi_1(0) = \varphi_1(2\pi)$
$\varphi_2(0) = \varphi_2(2\pi)$

$\Rightarrow \int_0^{2\pi}\varphi_n\varphi_m\,dx = 0$

$\therefore$ ~~$\varphi_n(0) \neq \varphi_n(2\pi)$~~

~~$\varphi_n'(0) \neq \varphi_n'(2\pi)$~~

Let $\varphi_1(x) = \cos nx$, $\varphi_2(x) = \sin nx$

$\varphi_1'(x) = -n\sin nx$, $\varphi_2'(x) = n\cos nx$

$\varphi_1''(x) = -n^2\cos nx$ $\varphi_2''(x) = -n^2\sin nx$

Also $\varphi_1(0) = \varphi_1(2\pi)$

$\varphi_2(0) = \varphi_2(2\pi)$

$\therefore \varphi_1'(0) = 0 = \varphi_1'(2\pi)$

$\varphi_2'(0) = \varphi_2'(2\pi)$

$\varphi_1(x) = \cos nx$, $\varphi_2(x) = \sin nx$ are two solns of the boundary.

5.b) Suppose φ_1, φ_2 are linearly independent solns of the constant coefficient eqn $y'' + a_1y' + a_2y = 0$ and let $W(\varphi_1, \varphi_2)$ be appreviated to W. Show that W is constant iff $a_1 = 0$

Soln:

Given φ_1 and φ_2 are two independence soln of

$y'' + a_1y' + a_2y = 0 \rightarrow (1)$

$(1) \Rightarrow \varphi_1'' + a_1\varphi_1' + a_2\varphi_1 = 0 \rightarrow (2)$

$\varphi_2'' + a_1\varphi_2' + a_2\varphi_2 = 0 \rightarrow (3)$

③ $\times\ \varphi_1 \Rightarrow \quad \varphi_1\varphi_2'' + a_1\varphi_1\varphi_2' + a_2\varphi_1\varphi_2 = 0$

② $\times\ \varphi_2 \Rightarrow \quad \varphi_2\varphi_1'' + a_1\varphi_2\varphi_1' + a_2\varphi_1\varphi_2 = 0$

$\quad (-) \quad (-) \quad (-)$

$$(\varphi_1\varphi_2'' - \varphi_2\varphi_1'') + a_1(\varphi_1\varphi_2' - \varphi_2\varphi_1') = 0 \rightarrow ④$$

But

$$w = \varphi_1\varphi_2' - \varphi_1'\varphi_2 \rightarrow ⑤$$

$$w' = \varphi_1\varphi_2'' + \varphi_1'\varphi_2' - \varphi_1''\varphi_2 - \varphi_1'\varphi_2'$$

$$w' = \varphi_1\varphi_2'' - \varphi_1''\varphi_2 \rightarrow ⑥$$

Sub ⑤ and ⑥ in ④

$$w' + a_1 w = 0 \rightarrow ⑦$$

Suppose $a_1 = 0$

⑦ $\Rightarrow w' = 0$

$w = a \Rightarrow$ constant

ie) $a_1 = 0 \Rightarrow w$ is constant $\rightarrow$ Ⓐ

Converse part:

Let w is constant

$w' = 0$

⑦ $\Rightarrow w' = -a_1 w$

$0 = -a_1 w$

$a_1 w = 0$

$\Rightarrow a_1 = 0$

ie) w is constant $\Rightarrow a_1 = 0 \rightarrow$ Ⓑ

By Ⓐ and Ⓑ we get,

w constant $\Leftrightarrow a_1 = 0$

Hence proved.

5.c) Consider the eqn $y'' + a_1 y' + a_2 y = 0$ where a_1, a_2 are real constants such that $4a_2 - a_1^2 > 0$. Let $\alpha + i\beta$, $\alpha - i\beta$. α, β be real root of the characteristic polynomial

(i) Show that φ_1, φ_2 defined by $\varphi_1(x) = e^{\alpha x}\cos\beta x$, $\varphi_2(x) = e^{\alpha x}\sin\beta x$ are the solns of the eqn

(ii) compute $W(\varphi_1, \varphi_2)$ and S.T φ_1, φ_2 are linearly independent on any interval I.

Soln:

Given equation $y''+a_1y'+a_2y=0 \rightarrow (1)$

The characteristic polynomial is

$$p(r)=r^2+a_1r+a_2=0$$

$$r=\frac{-a_1\pm\sqrt{a_1^2-4a_2}}{2}$$

$$=\frac{-a_1\pm\sqrt{-(4a_2-a_1^2)}}{2}$$

$$\alpha\pm i\beta=\frac{-a_1\pm i\sqrt{4a_2-a_1^2}}{2}$$

$$\left.\begin{aligned}\alpha+i\beta&=\frac{-a_1+i\sqrt{4a_2-a_1^2}}{2}\\ \alpha-i\beta&=\frac{-a_1-i\sqrt{4a_2-a_1^2}}{2}\end{aligned}\right\} \rightarrow (1)$$

Any soln φ is written of the form

$$\varphi(x)=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

$$\varphi(x)=A\varphi_1+B\varphi_2$$

Since $\varphi_1=e^{\alpha x}\cos\beta x$

$\varphi_2=e^{\alpha x}\sin\beta x$

(ii) To find $W(\varphi_1,\varphi_2)$

$\varphi_1(x)=e^{\alpha x}\cos\beta x$, $\varphi_2(x)=e^{\alpha x}\sin\beta x$

$\varphi_1'(x)=\alpha e^{\alpha x}\cos\beta x-\beta e^{\alpha x}\sin\beta x$, $\varphi_2'(x)=\alpha e^{\alpha x}\sin\beta x+\beta e^{\alpha x}\cos\beta x$

$$W(\varphi_1,\varphi_2)=\begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ \alpha e^{\alpha x}\cos\beta x-\beta e^{\alpha x}\sin\beta x & \alpha e^{\alpha x}\sin\beta x+\beta e^{\alpha x}\cos\beta x \end{vmatrix}$$

$$=\alpha e^{2\alpha x}\sin\beta x\cos\beta x+\beta e^{2\alpha}\cos^2\beta x-\alpha e^{2\alpha}\cos\beta x\sin\beta x+\beta e^{2\alpha}\sin^2\beta x$$

$$=\beta e^{2\alpha}(\cos^2\beta x+\sin^2\beta x)$$

$$=\beta e^{2\alpha}\neq 0$$

$\therefore$ The function φ_1,φ_2 are linearly independent on interval I

Hence proved.